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THE SCATTERING OF A PLANE WAVE BY A TRAP IN THE CRITICAL CASE†

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The scattering of a plane wave by a resonator with a narrow coupling channel is considered. The velocity potential of the scattered wave in this resonator has two series of poles with small imaginary parts, corresponding to the main trap and the coupling channel, the effect of which inside the trap differs by an order of magnitude. The critical case, when the limiting value for the poles from both series is the same, is investigated. It is shown that in this case two poles exist, which converge to this limiting value, and they both inherit resonance properties, characteristic for poles generated by the main trap. The principal terms of the asymptotic forms of the poles and the scattered wave are constructed. © 2002 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Suppose a space is filled with a uniform and isotropic liquid or gaseous medium. It is well known that in this case the velocity potential $U_\varepsilon(\mathbf{x}, \mathbf{k})$ of the scattered acoustic wave, which occurs when a plane wave $U_0(\mathbf{x}, \mathbf{k}) = e^{i(\mathbf{x},\mathbf{k})}$ is reflected from an ideal rigid body Ω^ε, is the solution of Neumann's problem

$$(\Delta + k^2)U_\varepsilon = 0, \quad \mathbf{x} \in \Omega_\varepsilon; \quad \frac{\partial U_\varepsilon}{\partial \mathbf{n}} = -\frac{\partial U_0}{\partial \mathbf{n}}, \quad \mathbf{x} \in \partial\Omega_\varepsilon$$
$$U_\varepsilon = O(r^{-1}), \quad \frac{\partial U_\varepsilon}{\partial r} - ikU_\varepsilon = o(r^{-1}), \quad r \to \infty \tag{1.1}$$

where

$$\Omega_\varepsilon = \mathbf{R}^3 \setminus \overline{\Omega^\varepsilon}, \quad \mathbf{x} = (x_1, x_2, x_3), \quad r = |\mathbf{x}|, \quad k = |\mathbf{k}|$$

$\mathbf{n}$ is the outward normal, while the complete wave in the region Ω^ε is defined by the equality

$$U^\varepsilon(\mathbf{x}, \mathbf{k}) = U_\varepsilon(\mathbf{x}, \mathbf{k}) + U_0(\mathbf{x}, \mathbf{k})$$

We will consider the case when Ω^ε is a trap – a region, homeomorphic to a spherical layer, in which a narrow coupling channel is cut (see Fig. 1). Suppose Ω^{in} and Ω are simply connected bounded regions in $\mathbf{R}^3$, $\overline{\Omega}^{\text{in}} \subset \Omega$, $\Omega^{\text{ex}} = \mathbf{R}^3 \setminus \overline{\Omega}$, $\partial\Omega^{\text{in(ex)}} \in C^\infty$. We will assume that Ω^{in} the neighbourhood of the origin of coordinates coincides with the half-space $x_3 > 0$, the region Ω^{ex} in the neighbourhood of the point $\mathbf{x}^{(0)} = (0, 0, -h)$, $h > 0$ coincides with the half-space $x_3 < -h$, while the section $[0, -h]$ on the Ox_3 axis does not contain points from $\Omega^{\text{in}} \cup \Omega^{\text{ex}}$. Further, suppose ω is a bounded region in the $x_3 = 0$ plane with a smooth boundary and $\omega_\varepsilon = \{\mathbf{x}: \mathbf{x}\varepsilon^{-1} \in \omega\}$, $0 < \varepsilon \ll 1$. The regions Ω^{in} and Ω^{ex} are the interior and exterior of the resonator $\Omega_\varepsilon = \Omega^{\text{in}} \cup \Omega^{\text{ex}} \cup \varkappa_\varepsilon$ respectively, where $\varkappa_\varepsilon = \omega_\varepsilon \times [0, -h]$ is the coupling channel. Boundary-value problem (1.1) will be called the perturbed problem, and the limiting internal (external) problem will be understood to be Neumann's boundary-value problem for Helmholtz' equations in the region Ω^{in} (in the region Ω^{ex}).

It is well known (see, for example, [1]), that for real k the perturbed problem and the limiting external problem are uniquely solvable, and their solutions allow of an analytic extension into the complex plane, which (for fixed ε) have a discrete set of poles Σ_ε and Σ^{ex} respectively, which lie below the real axis. On the other hand, it was shown in [2], that in Σ_ε there are two series of poles with small imaginary parts, the first of which, as $\varepsilon \to 0$, converges to the set Σ^{in} of natural frequencies (the roots of the eigenvalues)

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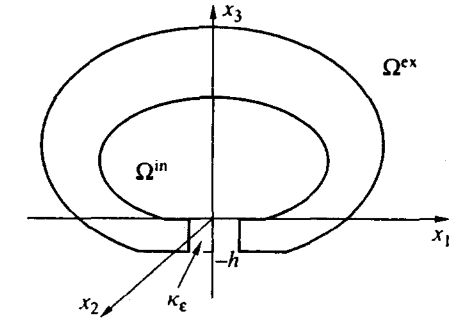

Fig. 1

of the limiting internal problem, while the second converges to the set $\Sigma^{ch} = \{m\pi / h\}_{m=1}^{\infty}$, generated by the presence of a coupling channel of finite length $h > 0$. It was shown in [3] that the poles, which converge to $k_0 \in \Sigma_1^{in} \setminus \Sigma^{ch}$, where Σ_1^{in} is the set of simple natural frequencies of the limiting internal problem, give rise to resonance phenomena in the scattering problem, which consists of the fact that for k close to k_0 the solution of problem (1.1) increases without limit in the region Ω^{in}. This effect has been called internal resonance.

It follows from results obtained previously [4, 5] that resonance phenomena are also observed at real frequencies close to zero and to $\Sigma^{ch} \setminus \Sigma^{in}$, in which case, it was shown in [5] that if in the region Ω^{ex} the qualitative behaviour of the solutions of problem (1.1) at frequencies close to $\Sigma^{ch} \setminus \Sigma^{in}$ and $\Sigma_1^{in} \setminus \Sigma^{ch}$ are the same: the solution of the perturbed problem differs from the solution of the limiting external problem by $O(1)$ (this effect will be called external resonance), then inside the trap (i.e. in the region Ω^{in}) the solution of the scattering problem at frequencies close to $k_0 \in \Sigma^{ch} \setminus \Sigma^{in}$ is bounded, and at frequencies close to $k_0 \in \Sigma_1^{in} \setminus \Sigma^{ch}$, it is of the order of ε^{-2}. Hence, in the first case there is no internal resonance. This difference in the behaviour of the solutions can be explained by the fact that, in the first case, the corresponding quasi-eigenfunction (the residue of the solution at the pole) is concentrated in $\varkappa_\varepsilon$, while in the second case it is concentrated in Ω^{in}. Below we investigate the effect of poles with a small imaginary part on the scattering of a plane wave on Ω^ε in the critical case, when the limiting value k_0 of these poles belongs to $\Sigma_1^{in} \cap \Sigma^{ch}$. Note that this situation arises for a fixed region Ω^{in} by changing the length h of the coupling channel.

2. FORMULATION OF THE RESULTS

Below we will construct asymptotic forms of the two poles $\tau_\varepsilon^{(1)}$ and $\tau_\varepsilon^{(2)}$, which converge to

$$k_0 = \pi m / h \in \Sigma^{ch} \cap \Sigma_1^{in} \tag{2.1}$$

when $\varepsilon \to 0$, and, in particular, we will show that

$$\tau_\varepsilon^{(n)} = k_0 + \varepsilon \tau_1^{(n)} + \varepsilon^2 \tau_2^{(n)} + \dots$$

where

$$\tau_1^{(n)} = -\frac{|\omega| \psi_0}{T^{(n)}}, \quad \operatorname{Im} \tau_2^{(n)} = -\frac{(k_0 |\omega| \psi_0)^2 \sigma(k_0)}{T^{(n)^2} + h |\omega| \psi_0^2} \tag{2.2}$$

$$T^{(n)} = k_0 q_0(\omega) + (-1)^n \sqrt{(k_0 q_0(\omega))^2 + h |\omega| \psi_0^2 / 2}, \quad \psi_0 = \psi(0)$$

$$\sigma(k) = \lim_{R \to \infty} \int_{r=R} |G^{ex}(\mathbf{x}, \mathbf{x}^{(0)}, k)|^2 \, ds > 0$$

$q_0(\omega)$ is a certain real constant, which depends only on the geometry of the region ω and will be determined in the next section (see formulae (3.27)), $\sigma(k)$ is the transverse section [6] of Green's function $G^{ex}(\mathbf{x}, \mathbf{x}^{(0)}, k)$ of the limiting external problem, and $\psi(x)$ is the eigenfunction of the limiting internal problem, normalized in $L_2(\Omega^{in})$, corresponding to the natural frequency k_0.

It follows from the definition of $T^{(n)}$ and expressions (2.2) that $\tau_1^{(1)} \neq \tau_2^{(2)}$, $\operatorname{Im} \tau_1^{(n)} = 0$ and $\operatorname{Im} \tau_2^{(n)} < 0$. Since the poles $\tau_\varepsilon^{(n)}$ of the analytic extension of the solution of problem (1.1) are situated at a distance $|\operatorname{Im} \tau_\varepsilon^{(n)}|$ from the real axis, while the solution itself can be considered for real frequencies k, it is obvious that the solution experiences the greatest effect of the pole for real frequencies $k^{(n)}(\varepsilon) = \operatorname{Re} \tau_\varepsilon^{(n)} + O(\operatorname{Im} \tau_\varepsilon^{(n)})$. These frequencies will be called peak frequencies and we will investigate the asymptotic form $U^\varepsilon(\mathbf{x}, \mathbf{k}^{(n)}(\varepsilon))$ when

$$\lim_{\varepsilon \to 0} \mathbf{k}^{(n)}(\varepsilon) = \mathbf{k}_0, \quad |\mathbf{k}^{(n)}(\varepsilon)| = k^{(n)}(\varepsilon)$$

It is obvious that $|\mathbf{k}_0| = k_0$. By virtue of relations (2.2) the peak frequencies are defined by the equations

$$k^{(n)}(\varepsilon) = \operatorname{Re} \tau_\varepsilon^{(n)} + \varepsilon^2 t \tag{2.3}$$

where t is an arbitrary real number.

We will denote by $U^0(\mathbf{x}, \mathbf{k})$ the complete wave which occurs when a plane wave $U_0(\mathbf{x}, \mathbf{k})$ is scattered by Ω (i.e. the sum $U_0(\mathbf{x}, \mathbf{k})$ and the solutions of the limiting external problem). We will show that if relation (2.1) is satisfied, then, as $\varepsilon \to 0$

$$\begin{aligned} &U^\varepsilon(\mathbf{x}; \mathbf{k}^{(n)}(\varepsilon)) \sim \varepsilon^{-1} C^{(n)}(t) T^{(n)} \psi(\mathbf{x}) / \psi_0, \quad \mathbf{x} \in \Omega^{\text{in}} \\ &U^\varepsilon(\mathbf{x}; \mathbf{k}^{(n)}(\varepsilon)) \sim \varepsilon^{-2} C^{(n)}(t) \sin(k_0 x_3), \quad \mathbf{x} \in \varkappa_\varepsilon \\ &U^\varepsilon(\mathbf{x}; \mathbf{k}^{(n)}(\varepsilon)) \sim (-1)^m k_0 |\omega| C^{(n)}(t) G^{\text{ex}}(\mathbf{x}, \mathbf{x}^{(0)}, k_0) + U^0(\mathbf{x}; \mathbf{k}_0), \quad \mathbf{x} \in \Omega^{\text{ex}} \end{aligned} \tag{2.4}$$

where

$$C^{(n)}(t) = \frac{(-1)^m k_0 |\omega| \psi_0^2 U^0(\mathbf{x}_0, \mathbf{k}_0)}{2k_0(i\tilde{\tau}_2^{(n)} - t)(T^{(n)^2} + h|\omega|\psi_0^2/2)}, \quad \tilde{\tau}_2^{(n)} = \operatorname{Im} \tau_2^{(n)}$$

It follows from relations (2.4) that in both cases (i.e. for $n = 1$ and $n = 2$) at the peak frequencies both external and internal resonance is observed.

3. CONSTRUCTION OF THE ASYMPTOTIC FORMS OF THE POLES

Consider the boundary-value problem with a source

$$\begin{aligned} &(\Delta + k^2) u_\varepsilon = F, \quad \mathbf{x} \in \Omega_\varepsilon; \quad \partial u_\varepsilon / \partial \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega_\varepsilon \\ &u_\varepsilon = O(r^{-1}), \quad \partial u_\varepsilon / \partial r - iku_\varepsilon = o(r^{-1}), \quad r \to \infty \end{aligned} \tag{3.1}$$

Following the approach described previously [4] and bearing in mind that, as is well known [7], the joint multiplicity of the residues at the poles, which converge to $k_0 \in \Sigma_1^{\text{in}} \cap \Sigma^{\text{ch}}$, is equal to two, it can be shown that in this case the analytic extension of the solution of boundary-value problem (3.1), with k close to k_0, has the form

$$u_\varepsilon(\mathbf{x}, \mathbf{k}) = \sum_{n=1}^{2} \frac{\Psi_\varepsilon^{(n)}(\mathbf{x})}{k^2 - \tau_\varepsilon^{(n)^2}} \int_{\Omega_\varepsilon} F(\mathbf{y}) \Psi_\varepsilon^{(n)}(\mathbf{y}) dy + \tilde{u}_\varepsilon(\mathbf{x}, \mathbf{k}) \tag{3.2}$$

where, when $\varepsilon \to 0$, the function $\tilde{u}_\varepsilon$ is bounded, and if moreover, $\operatorname{supp} F \subset \Omega^{\text{ex}}$, then $\tilde{u}_\varepsilon$ converges to the solution u_0 of the limiting external problem in Ω^{ex} and to zero outside Ω^{ex} (with respect to the norm L_2 on any compactum). The quasi-eigenfunctions $\Psi_\varepsilon^{(n)}$ for fixed ε satisfy the equations

$$(\Delta + \tau_\varepsilon^{(n)^2}) \Psi_\varepsilon^{(n)} = 0 \quad \text{in} \quad \Omega_\varepsilon$$

the Neumann homogeneous boundary condition on $\partial\Omega_\varepsilon$ and increase exponentially at infinity, and when $\varepsilon \to 0$

$$\Psi_\varepsilon^{(n)}(\mathbf{x}) \to 0 \quad \text{in} \quad \Omega^{ex}$$

(with respect to the norm L_2 in any compactum) and

$$\begin{gathered}\Psi_\varepsilon^{(n)}(\mathbf{x}) \to \alpha^{(n)}\psi(\mathbf{x}) \quad \text{in} \quad \Omega^{in} \\ \Psi_\varepsilon^{(n)}(\mathbf{x}) - \varepsilon^{-1}\beta^{(n)}\sqrt{\frac{2}{h|\omega|}}\sin(k_0 x_n) \to 0 \quad \text{in} \quad \varkappa_\varepsilon\end{gathered} \tag{3.3}$$

where $\alpha^{(n)}$ and $\beta^{(n)}$ are certain real numbers, normalized by the equation

$$\alpha^{(n)^2} + \beta^{(n)^2} = 1 \tag{3.4}$$

Remarks 1. Conditions (3.3) and (3.4) indicate that when $\varepsilon \to 0$ the norm $\Psi_\varepsilon^{(n)}$ in $L_2(\Omega^{in} \cup \varkappa_\varepsilon)$ tends to unity. The form of the principal terms of the asymptotic forms (3.3) itself is a linear combination of the principal terms of the asymptotic forms of the quasi-eigenfunctions, corresponding to the cases $k_0 \in \Sigma_1^{in}\backslash\Sigma^{ch}$ and $k_0 \in \Sigma^{ch}\backslash\Sigma^{in}$ considered earlier in [5]. In both these cases one quasi-eigenfunction exists with respect to one pole (i.e. instead of the singular sum with respect to n on the right-hand side of relation (3.2) there is only one singular term), but the values $\alpha = 1$ and $\beta = 0$ correspond to the first case, and $\alpha = 0$ and $\beta = 1$ correspond to the second case in relations (3.3).

2. The problem of the scattering of a plane wave clearly reduces to boundary-value problem (3.1). In turn, in order to obtain the principal terms of the asymptotic forms of the solution of problem (3.1) from representation (3.2), it is sufficient to know the quantities $\alpha^{(n)}$ and $\beta^{(n)}$ and the principal (non-zero) terms of the asymptotic form Im $\tau_\varepsilon^{(n)}$ and $\Psi_\varepsilon^{(n)}$ in Ω^{ex}. The determination of the values of these parameters is also the main purpose of the present section.

We will denote Green's function of the limiting internal problem by $G^{in}(\mathbf{x}, \mathbf{y}, k)$ and put

$$\begin{gathered}\psi_\varepsilon^{in}(\mathbf{x}, k) = (k_0^2 - k^2)(a_0 + \varepsilon L_1^{in}(D_y) + \varepsilon^2 L_2^{in}(D_y))G^{in}(\mathbf{x}, \mathbf{y}, k)|_{\mathbf{y}=0} \\ \psi_\varepsilon^{ex}(\mathbf{x}, k) = (\varepsilon b_1 + \varepsilon^2 L_1^{ex}(D_y))G^{ex}(\mathbf{x}, \mathbf{y}, k)|_{\mathbf{y}=\mathbf{x}_0} \\ L_1^{in}(D_y) = \sum_{q=1}^{2} a_{1q}\frac{\partial}{\partial y_q}, \quad L_2^{in}(D_y) = \sum_{j=1}^{2}\sum_{q=1}^{j} a_{2jq}\frac{\partial^2}{\partial y_j\,\partial y_q} + \sum_{q=1}^{2} a_{2q}\frac{\partial}{\partial y_q} \\ L_1^{ex}(D_y) = b_2 + \sum_{q=2}^{2} b_{2q}\frac{\partial}{\partial y_q} \\ \psi_\varepsilon^{ch}(\mathbf{x}) = \varepsilon^{-1}w_{-1}(x_3) + w_0(x_3) + \varepsilon w_1(x_3)\end{gathered} \tag{3.5}$$

where

$$w_{-1}(t) = c_{-1}\sin(k_0 t) \tag{3.6}$$

and a_0, a_{jm}, a_{2jm}, b_{jm}, c_{-1} and $w_0(t)$ and $w_1(t)$ are, for the present, arbitrary constants and functions respectively. By definition the function $\psi_\varepsilon^{in}(\mathbf{x}, k)$ (the function $\psi_\varepsilon^{ex}(\mathbf{x}, k)$ satisfies the equation $(\Delta + k^2)\psi_\varepsilon^{in} = 0$ in the region Ω^{in} (the equation $(\Delta + k^2)\psi_\varepsilon^{ex} = 0$ in the region Ω^{ex}) and the Neumann homogeneous boundary condition on $\partial\Omega^{in}\backslash\{0\}$ (on $\partial\Omega^{ex}\backslash\{\mathbf{x}_0\}$) and

$$\psi_\varepsilon^{in}(\mathbf{x}, k) = a_0\psi_0\psi(\mathbf{x}) + o(1) \quad \text{as} \quad k \to k_0 \quad \text{in} \quad \overline{\Omega^{in}} \setminus \{0\} \tag{3.7}$$

The principal terms of the asymptotic forms of the poles and of the corresponding quasi-eigenfunctions will be sought in the form

$$\tau_\varepsilon \approx k_0 + \varepsilon\tau_1 + \varepsilon^2\tau_2 \tag{3.8}$$

$$\Psi_\varepsilon(\mathbf{x}) \approx \psi_\varepsilon^{\text{in}}(\mathbf{x}, \tau_\varepsilon) \quad \text{in} \quad \Omega^{\text{in}} \setminus S^{\text{in}}(\varepsilon^{1/2}) \tag{3.9}$$

$$\Psi_\varepsilon(\mathbf{x}) \approx \psi_\varepsilon^{\text{ex}}(\mathbf{x}, \tau_\varepsilon) \quad \text{in} \quad \Omega^{\text{ex}} \setminus S^{\text{ex}}(\varepsilon^{1/2}) \tag{3.10}$$

$$\Psi_\varepsilon(\mathbf{x}) \approx \psi_\varepsilon^{\text{ch}}(\mathbf{x}) \quad \text{in} \quad \varkappa_\varepsilon \setminus (S^{\text{ex}}(\varepsilon^{1/2}) \cup S^{\text{in}}(\varepsilon^{1/2})) \tag{3.11}$$

Here and henceforth $S^{\text{in}}(r)$ and $S^{\text{ex}}(r)$ are spheres of radius R with centres at 0 and $\mathbf{x}_0$ respectively, while the subscript n of the corresponding nth pole will henceforth be omitted for brevity (wherever possible). It follows from relations (3.5)–(3.9) and (3.11) that the normalization conditions (3.3) and (3.4) have the form

$$(a_0\psi_0)^2 + \tfrac{1}{2}c_{-1}^2 h\,|\,\omega\,| = 1 \tag{3.12}$$

Hence, we have obtained the first equation for the coefficients a_0 and c_{-1}.

The function (3.5) obviously satisfies the Neumann homogeneous boundary condition on the walls of the coupling channel $\varkappa_\varepsilon$. Substituting relations (3.8), (3.11) and (3.5) into the equation $(\Delta + \tau_\varepsilon^2)\Psi\varepsilon = 0$ with $\mathbf{x} \in \varkappa_\varepsilon$, we obtain the following equations for the coefficients w_j

$$w_j''(x_3) + k_0^2 w_j(x_3) + \sum_{i=1}^{j+1} \lambda_i w_{j-i}(x_3) = 0, \quad -h < x_3 < 0$$

where

$$\lambda_1 = 2k_0\tau_1, \quad \lambda_2 = \tau_1^2 + 2k_0\tau_2 \tag{3.13}$$

It is easy to see that the solutions of these equations are the function (3.6) and the functions

$$\begin{aligned} w_0(x_3) &= c_{-1}\tau_1 x_3\cos(k_0x_3) + c_0\cos(k_0x_3) + A\sin(k_0x_3) \\ w_1(x_3) &= c_{-1}\left(-\frac{1}{2}\tau_1^2 x_3^2\sin(k_0x_3) + \tau_2 x_3\cos(k_0x_3)\right) - \tau_1 c_0 x_3\sin(k_0x_3) + \\ &+ A\tau_1 x_3\cos(k_0x_3) + c_1\cos(k_0x_3) + B\sin(k_0x_3) \end{aligned} \tag{3.14}$$

for any constants c_0, c_1, A and B.

The unknown constants a_j, b_j and c_j will be determined by the method of matched asymptotic expansions [8], by introducing inner expansions in the neighbourhood of the ends of the coupling channel $\varkappa_\varepsilon$ (in $S^{\text{in}}(2\varepsilon^{1/2})\cap\Omega_\varepsilon$ and $S^{\text{ex}}(2\varepsilon^{1/2})\cap\Omega_\varepsilon$) and matching them with expansions (3.9) and (3.11) at one end of the channel and with expansions (3.10) and (3.11) at the other end of the channel. It follows from the definition of $\psi_\varepsilon^{\text{in}}$ and $\psi_\varepsilon^{\text{ex}}$, the asymptotic form of the function G^{in} at zero and of the function G^{ex} when $|\mathbf{x} - \mathbf{x}_0| \to 0$ (see, for example, [5]), that

$$\begin{aligned} \psi_\varepsilon^{\text{in}}(\mathbf{x}, k) &= a_0\left(\psi_0\left(\psi_0 + \sum_{q=1}^{2}\psi_q x_q\right) + (k_0^2 - k^2)\left(\frac{1}{2\pi r} + g^{\text{in}}\right)\right) + \\ &+ \varepsilon\left(\psi_0\sum_{q=1}^{2} a_{1q}\psi_q - \frac{k_0^2 - k^2}{2\pi}L_1^{\text{in}}(D_x)\frac{1}{r}\right) + \varepsilon^2\frac{k_0^2 - k^2}{2\pi}\tilde{L}_2^{\text{in}}(D_x)\frac{1}{r} + \\ &+ O((r + \varepsilon + |k - k_0|)(r + \varepsilon)) \quad \text{when} \quad k \to k_0, \quad \mathbf{x} \to 0, \quad \varepsilon \to 0 \end{aligned} \tag{3.15}$$

$$\begin{aligned} \psi_\varepsilon^{\text{ex}}(\mathbf{x}, k) &= \varepsilon b_1\left(\frac{1}{2\pi|\mathbf{x} - \mathbf{x}_0|} + g^{\text{ex}}\right) + \varepsilon^2\frac{1}{2\pi}\tilde{L}_1^{\text{ex}}(D_x)\frac{1}{|\mathbf{x} - \mathbf{x}_0|} + \\ &+ O(\varepsilon|\mathbf{x} - \mathbf{x}_0| + \varepsilon^2) \quad \text{when} \quad k \to k_0, \quad \mathbf{x} \to \mathbf{x}_0, \quad \varepsilon \to 0 \end{aligned} \tag{3.16}$$

where

$$\tilde{L}_2^{\text{in}}(D_x)=\sum_{j=1}^{2}\sum_{q=1}^{j}a_{2jq}\frac{\partial^2}{\partial x_j\partial x_q}-\sum_{q=1}^{2}a_{2q}\frac{\partial}{\partial x_q},\quad \tilde{L}_1^{\text{ex}}(D_x)=b_2-\sum_{q=1}^{2}b_{2q}\frac{\partial}{\partial x_q}$$

$$g^{\text{in}}=\lim_{k\to k_0}\left(G^{\text{in}}(\mathbf{x},0,k)-\frac{\psi_0^2}{k_0^2-k^2}-\frac{1}{2\pi r}\right)\Bigg|_{\mathbf{x}=0}$$

$$g^{\text{ex}}=\left(G^{\text{ex}}(\mathbf{x},\mathbf{x}_0,k_0)-\frac{1}{2\pi|\mathbf{x}-\mathbf{x}_0|}\right)\Bigg|_{\mathbf{x}=\mathbf{x}_0}$$

Note that

$$\operatorname{Im} g^{\text{in}}=0,\quad \operatorname{Im} g^{\text{ex}}=k_0\sigma(k_0) \tag{3.17}$$

The first of these equations follows immediately from the fact that Green's function of the limiting internal problem is real for real k^2, while the second equality is well known (see, for example, [9]).

Rewriting the asymptotic forms $\psi_\varepsilon^{\text{in}}(\mathbf{x},\tau_\varepsilon)$ and $\psi_\varepsilon^{\text{ch}}(\mathbf{x})$ as $\mathbf{x}\to 0$ in the inner variables $\mathbf{X}=\mathbf{x}\varepsilon^{-1}$, and taking relations (3.8), (3.15) and (3.5), (3.6) and (3.14) into account, we obtain that when $\varepsilon^{1/2}<r<2\varepsilon^{1/2}$ (or, which is the same thing, when $\varepsilon^{-1/2}<\rho=|\mathbf{X}|<2\varepsilon^{-1/2}$)

$$\begin{aligned}\psi_\varepsilon^{\text{in}}(\mathbf{x},\tau_\varepsilon)&=V_0^{\text{in}}(\mathbf{X})+\varepsilon V_1^{\text{in}}(\mathbf{X})+O(\varepsilon^2\rho^2)\\ \psi_\varepsilon^{\text{ch}}(\mathbf{x})&=W_0^{\text{in}}(\mathbf{X})+\varepsilon W_1^{\text{in}}(\mathbf{X})+O(\varepsilon^2X_3^3)\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}V_0^{\text{in}}(\mathbf{X})&=a_0\psi_0^2+\frac{k_0\tau_1}{\pi}\left(-a_0+\sum_{q=1}^{2}a_{1q}\frac{\partial}{\partial X_q}-\sum_{j=1}^{2}\sum_{q=1}^{j}a_{2jq}\frac{\partial^2}{\partial X_j\partial X_q}\right)\frac{1}{\rho}\\ V_1^{\text{in}}(\mathbf{X})&=a_0\psi_0\sum_{q=1}^{2}\psi_qX_q+\left(\psi_0\sum_{q=1}^{2}a_{1q}\psi_q-a_02k_0\tau_1g^{\text{in}}\right)-\\ &-\frac{1}{2\pi}\left(a_0(\tau_1^2-2k_0\tau_2)+\sum_{q=1}^{2}((\tau_1^2+2k_0\tau_2)a_{1q}+2k_0\tau_1a_{2q})\frac{\partial}{\partial X_q}\right)\frac{1}{\rho}\\ W_0^{\text{in}}(\mathbf{X})&=c_{-1}k_0X_3+c_0\\ W_1^{\text{in}}(\mathbf{X})&=(c_{-1}\tau_1+Ak_0)X_3+c_1\end{aligned} \tag{3.19}$$

Similarly, rewriting the asymptotic forms $\psi_\varepsilon^{\text{ex}}(\mathbf{x},\tau_\varepsilon)$ and $\psi_\varepsilon^{\text{ch}}(\mathbf{x})$ as $\mathbf{x}\to\mathbf{x}_0$ in the variables $\mathbf{X}=(\mathbf{x}_0-\mathbf{x})\varepsilon^{-1}$, and taking relations (3.8), (3.16) and (3.5), (3.6) and (3.14) into account we obtain that when $\varepsilon^{1/2}<|\mathbf{x}-\mathbf{x}_0|<2\varepsilon^{1/2}$ (or, which is the same thing, when $\varepsilon^{-1/2}<\rho<2\varepsilon^{-1/2}$)

$$\begin{aligned}\psi_\varepsilon^{\text{ex}}(\mathbf{x},\tau_\varepsilon)&=V_0^{\text{ex}}(\mathbf{X})+\varepsilon V_1^{\text{ex}}(\mathbf{X})+O(\varepsilon^2\rho^2)\\ \psi_\varepsilon^{\text{ch}}(\mathbf{x})&=W_0^{\text{ex}}(\mathbf{X})+\varepsilon W_1^{\text{ex}}(\mathbf{X})+O(\varepsilon^2X_3^3)\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}V_0^{\text{ex}}(\mathbf{X})&=\frac{1}{2\pi}\left(b_1-\sum_{q=1}^{2}b_{2q}\frac{\partial}{\partial X_q}\right)\frac{1}{\rho}\\ V_1^{\text{ex}}(\mathbf{X})&=b_1g^{\text{ex}}+\frac{b_2}{2\pi}\frac{1}{\rho}\\ W_0^{\text{ex}}(\mathbf{X})&=(-1)^{m+1}(c_{-1}k_0X_3+c_{-1}\tau_1h-c_0)\\ W_1^{\text{ex}}(\mathbf{X})&=(-1)^{m+1}((c_{-1}\tau_1+Ak_0)X_3+c_{-1}\tau_2h+A\tau_1h-c_1)\end{aligned} \tag{3.21}$$

Bearing Eqs (3.18) and (3.20) in mind, and following the method of matched asymptotic expansions, in the neighbourhood of the ends of the coupling channel we seek the asymptotic forms of the quasi-eigenfunctions in the form

$$\Psi_\varepsilon^{(n)}(\mathbf{x}) = \upsilon_0^{\text{in}}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon\,\upsilon_1^{\text{in}}\left(\frac{\mathbf{x}}{\varepsilon}\right), \quad \mathbf{x} \in \Omega_\varepsilon \cap S^{\text{in}}(2\varepsilon^{1/2}) \tag{3.22}$$

$$\Psi_\varepsilon^{(n)}(\mathbf{x}) = \upsilon_0^{\text{ex}}\left(\frac{\mathbf{x}_0 - \mathbf{x}}{\varepsilon}\right) + \varepsilon\,\upsilon_1^{\text{ex}}\left(\frac{\mathbf{x}_0 - \mathbf{x}}{\varepsilon}\right), \quad \mathbf{x} \in \Omega_\varepsilon \cap S^{\text{ex}}(2\varepsilon^{1/2}) \tag{3.23}$$

where

$$\begin{aligned} &\upsilon_j^{\text{in}}(\mathbf{X}) = V_j^{\text{in}}(\mathbf{X}) + O(\rho^{-4+j}), \quad \upsilon_j^{\text{ex}}(\mathbf{X}) = V_j^{\text{ex}}(\mathbf{X}) + O(\rho^{-3+j}), \quad X_3 \geqslant 0 \\ &\upsilon_j^{\text{in}}(\mathbf{X}) = W_j^{\text{in}}(\mathbf{X}) + o(1), \quad \upsilon_j^{\text{ex}}(\mathbf{X}) = V_j^{\text{ex}}(\mathbf{X}) + o(1), \quad X_3 < 0 \end{aligned} \tag{3.24}$$

when $\rho \to \infty$.

Substituting expressions (3.8) and (3.2) (expressions (3.8) and (3.23)) into the equation $(\Delta + \tau_\varepsilon^2)\Psi_\varepsilon^{(n)} = 0$ in Ω_ε, requiring that the homogeneous boundary conditions for the functions (3.22) (for the function (3.23)) be satisfied on $\partial\Omega_\varepsilon$, and changing to the inner variables $\mathbf{X} = \mathbf{x}\varepsilon^{-1}$ (to the inner variable $\mathbf{X} = (\mathbf{x}_0 - \mathbf{x})\varepsilon^{-1}$), we obtain the boundary-value problems for υ_j^{in} (for υ_j^{ex})

$$\Delta\upsilon_j^{\text{in(ex)}} = 0, \quad \mathbf{X} \in \gamma_\omega; \quad \frac{\partial}{\partial\mathbf{n}}\upsilon_j^{\text{in(ex)}} = 0, \quad \mathbf{X} \in \partial\gamma_\omega \tag{3.25}$$

where

$$\gamma_\omega = \{\mathbf{X} : X_3 > 0\} \cup \{\mathbf{X} : (X_1, X_2) \in \omega, \ -\infty < X_3 \leqslant 0\}$$

It was shown in [4] that solutions Z_q of boundary-value problem (3.25) exist having the asymptotic forms

$$\begin{aligned} &Z_0(\mathbf{X}) = \left(-\frac{|\omega|}{2\pi} + \sum_{t=1}^{2} d_{0t}(\omega)\frac{\partial}{\partial X_t} + \sum_{t=1}^{2}\sum_{s=1}^{t} d_{0ts}(\omega)\frac{\partial^2}{\partial X_t \partial X_s}\right)\frac{1}{\rho} + O(\rho^{-4}) \\ &Z_j(\mathbf{X}) = X_j + \sum_{t=1}^{2} d_{jt}(\omega)\frac{\partial}{\partial X_t}\left(\frac{1}{\rho}\right) + O(\rho^{-3}), \quad j = 1, 2 \end{aligned} \tag{3.26}$$

when $\rho \to \infty$, $X_3 \geqslant 0$, and the asymptotic forms

$$Z_0(\mathbf{X}) = X_3 + q_0(\omega) + O(e^{\mu X_3}), \quad Z_j(\mathbf{X}) = q_j(\omega) + O(e^{\mu X_3}), \quad j = 1, 2 \tag{3.27}$$

when $\rho \to \infty$, $X_3 < 0$, where $\mu > 0$ is the second natural frequency of the two-dimensional Neumann problem for $-\Delta$ in ω, while the coefficients $d_{i,t}$, $d_{i,t,s}$, $q_i(\omega)$ depend on the geometry of the region ω.

We put

$$\begin{aligned} &\upsilon_0^{\text{in}}(\mathbf{X}) = c_{-1}k_0 Z_0(\mathbf{X}) + a_0\psi_0^2 \\ &\upsilon_1^{\text{in}}(\mathbf{X}) = (c_{-1}\tau_1 + Ak_0)Z_0(\mathbf{X}) + a_0\psi_0\sum_{q=1}^{2}\psi_q Z_q(\mathbf{X}) + \left(\psi_0\sum_{q=1}^{2} a_{1q}\psi_q - a_0 2k_0\tau_1 g^{\text{in}}\right) \\ &\upsilon_0^{\text{ex}}(\mathbf{X}) = (-1)^{m+1}c_{-1}k_0 Z_0(\mathbf{X}) \\ &\upsilon_1^{\text{ex}}(\mathbf{X}) = (-1)^{m+1}(c_{-1}\tau_1 + Ak_0)Z_0(\mathbf{X}) + b_1 g^{\text{ex}} \end{aligned} \tag{3.28}$$

It follows from relations (3.19), (3.21) and (3.26)–(3.28) that the necessary and sufficient conditions for Eqs (3.24) to be satisfied are the equations

$$c_{-1}k_0 q_0(\omega) + a_0\Psi_0^2 = c_0, \quad c_{-1}|\omega|/2 = \tau_1 a_0 \tag{3.29}$$

$$(c_{-1}\tau_1 + Ak_0)q_0(\omega) + \psi_0 \sum_{j=1}^{2} (q_j(\omega)a_0 + a_{1j})\psi_j - 2a_0k_0\tau_1 g^{\text{in}} = c_1 \tag{3.30}$$

$$(c_{-1}\tau_1 + Ak_0)\,|\,\omega\,| = a_0(\tau_1^2 + 2k_0\tau_2) \tag{3.31}$$

$$c_{-1}k_0 q_0(\omega) = c_{-1}\tau_1 h - c_0, \quad (-1)^m c_{-1}k_0\,|\,\omega\,| = b_1 \tag{3.32}$$

$$(c_{-1}\tau_1 + Ak_0)q_0(\omega) + (-1)^{m+1} b_1 g^{\text{ex}} = c_{-1}\tau_2 h + A\tau_1 h - c_1 \tag{3.33}$$

$$c_{-1}d_{0t}(\omega) = \frac{\tau_1}{\pi} a_{1t}, \quad c_{-1}d_{0ts}(\omega) = -\frac{\tau_1}{\pi} a_{2ts} \tag{3.34}$$

$$(c_{-1}\tau_1 + Ak_0)d_{0t}(\omega) + a_0\psi_0 \sum_{q=1}^{2} \psi_q d_{qt}(\omega) = \frac{1}{2\pi}((\tau_1^2 + 2k_0\tau_2)a_{1t} + 2k_0\tau_1 a_{2t}) \tag{3.35}$$

$$(-1)^m (c_{-1}\tau_1 + Ak_0)\,|\,\omega\,| = b_2 \tag{3.36}$$

Solving Eqs (3.12), (3.29) and (3.32), we obtain two series of values of $a_0^{(n)}$, $c_{-1}^{(n)}$, $\tau_1^{(n)}$, $b_1^{(n)}$, $c_1^{(n)}$ $(n = 1, 2)$ and, in particular, we obtain the formulae

$$a_0^{(n)} = -\frac{T^{(n)}}{\psi_0 T_n}, \quad c_{-1}^{(n)} = \frac{\psi_0}{T_n}, \quad b_1^{(n)} = (-1)^m \frac{k_0\,|\,\omega\,|\,\psi_0}{T_n} \tag{3.37}$$

$$T_n = \sqrt{T^{(n)^2} + h\,|\,\omega\,|\,\psi_0^2/2}$$

and (2.2) for $\tau_1^{(n)}$. Solving Eqs (3.34) and (3.36) we obtain the quantities $a_{1t}^{(n)}$, $a_{2ts}^{(n)}$ and $b_2^{(n)}$. Finally, from Eqs (3.13), (3.31), (3.33) and (3.35) we determine $A^{(n)}$, $\tau_2^{(n)}$, $c_1^{(n)}$, $a_{2t}^{(n)}$ and, in particular, we obtain that

$$\operatorname{Im}\tau_2^{(n)} = -\frac{k_0\,|\,\omega\,|^2\,\psi_0^2}{T^{(n)^2} + h\,|\,\omega\,|\,\psi_0^2}\operatorname{Im} g^{\text{ex}} \tag{3.38}$$

The quantity $\operatorname{Im}\tau_\varepsilon^{(n)}$ in formulae (2.2) follows from Eqs (3.38) and (3.17).

We emphasize that by choosing the coefficients $a_0^{(n)}$, $b_j^{(n)}$, $c_q^{(n)}$, $a_{jt}^{(n)}$, $a_{2ts}^{(n)}$ and $A^{(n)}$ which satisfy Eqs (3.29)–(3.36), we attempted to satisfy Eqs (3.24). Hence, the asymptotic expansions (3.9) and (3.11) were (in principal terms) matched with the function (3.22) in $S^{\text{in}}(2\varepsilon^{1/2}) \setminus S^{\text{in}}(\varepsilon^{3/2})$, and the asymptotic expansions (3.10) and (3.11) with the function (3.23) in $S^{\text{ex}}(2\varepsilon^{1/2}) \setminus S^{\text{ex}}(\varepsilon^{1/2})$. It follows from relations (3.5)–(3.11) and (3.37) that

$$\begin{aligned}
&\Psi_\varepsilon^{(n)}(\mathbf{x}) \approx c_{-1}^{(n)} T^{(n)}\psi(\mathbf{x})/\psi_0 \quad \text{in} \quad \Omega^{\text{in}} \setminus S^{\text{in}}(\varepsilon^{1/2})\\
&\Psi_\varepsilon^{(n)}(\mathbf{x}) \approx \varepsilon(-1)^m k_0\,|\,\omega\,|\,c_{-1}^{(n)} G^{\text{ex}}(\mathbf{x}, \mathbf{x}_0, k_0) \quad \text{in} \quad \Omega^{\text{ex}} \setminus S^{\text{ex}}(\varepsilon^{1/2})\\
&\Psi_\varepsilon^{(n)}(\mathbf{x}) \approx \varepsilon^{-1} c_{-1}^{(n)} \sin(k_0 x_3) \quad \text{in} \quad \varkappa_\varepsilon \setminus (S^{\text{ex}}(\varepsilon^{1/2}) \cup S^{\text{in}}(\varepsilon^{1/2}))
\end{aligned} \tag{3.39}$$

4. THE PRINCIPAL TERMS OF THE ASYMPTOTIC FORMS OF THE SOLUTIONS FOR SCATTERING PROBLEMS

It follows from relations (3.39) and (3.2) that for the peak frequencies (2.3) the principal terms of the asymptotic forms of the solution of problem (3.10 (i.e. the problem with an external source) have the form

$$\begin{aligned}
&u_\varepsilon(\mathbf{x}; \mathbf{k}^{(n)}(\varepsilon)) \sim \varepsilon^{-1} c^{(n)}(t) T^{(n)} \psi(\mathbf{x})/\psi_0, \quad \mathbf{x} \in \Omega^{\text{in}}\\
&u_\varepsilon(\mathbf{x}; \mathbf{k}^{(n)}(\varepsilon)) \sim \varepsilon^{-2} c^{(n)}(t) \sin(k_0 x_3), \quad \mathbf{x} \in \varkappa_\varepsilon\\
&u_\varepsilon(\mathbf{x}; \mathbf{k}^{(n)}(\varepsilon)) \sim (-1)^m k_0\,|\,\omega\,|\,c^{(n)}(t) G^{\text{ex}}(\mathbf{x}, \mathbf{x}^{(0)}, k_0) + u_0(\mathbf{x}; \mathbf{k}_0), \quad \mathbf{x} \in \Omega^{\text{ex}}
\end{aligned} \tag{4.1}$$

where

$$c^{(n)}(t) = \frac{(-1)^m k_0 |\omega| (c_{-1}^{(n)})^2 u_0(\mathbf{x}_0, \mathbf{k}_0)}{2k_0(i\tilde{\tau}_2^{(n)} - t)}, \quad \tilde{\tau}_2^{(n)} = \operatorname{Im} \tau_2^{(n)}$$

Suppose $\chi(t)$ is an infinitely differentiable shear function, identically equal to unity when $t < 1$ and zero when $t > 2$; $L > 0$ is a fairly large number such that $\overline{\Omega} \subset S(L)$, $\delta > 0$. It is obvious that the function U^δ can also be represented in the form

$$U^\delta(\mathbf{x}, \mathbf{k}) = U_0(\mathbf{x}, \mathbf{k})(1 - \chi(rL^{-1})) + u_\delta(\mathbf{x}, \mathbf{k}) \tag{4.2}$$

where u_ε is the solution of problem (3.1), and u_0 is the solution of the limiting external problem in Ω^{ex} with right-hand sides equal to

$$F = U^0 \Delta\chi + 2\sum_{i=1}^{3} \frac{\partial \chi}{\partial x_i} \frac{\partial U_0}{\partial x_i}$$

Since supp $F \in \Omega^{ex}$, then u_ε and u_0 are solutions of the problems with an external source.

It follows from relations (4.1), in particular, that

$$U^0(\mathbf{x}, \mathbf{k}) = U_0(\mathbf{x}, \mathbf{k})(1 - \chi(rL^{-1})) + u_0(\mathbf{x}, \mathbf{k}), \quad u_0(\mathbf{x}_0, \mathbf{k}_0) = U^0(\mathbf{x}_0, \mathbf{k}_0) \tag{4.3}$$

Substituting relations (4.1) into expansion (4.2) and bearing in mind equality (4.3), we obtain formulae (2.4) of the principal terms of the asymptotic forms of the complete wave for peak frequencies.

5. CONCLUDING REMARKS

In Section 3, using the method of matched asymptotic expansions, we constructed the first terms of the asymptotic forms of the poles and the corresponding quasi-eigenfunctions. The remaining terms of the asymptotic forms (i.e. the complete formal asymptotic expansion) are constructed similarly. In particular, when matching the following terms of the expansion one determines the constants b_{2q} in the definition of ψ^{ex} and B in expressions (3.14). These terms are introduced in order to verify that they have no effect on the equations for determining the principal parameters τ_j, a_0, b_1 and c_{-1}. For the cases $k_0 \in \Sigma_1^{in} \backslash \Sigma^{ch}$ and $k_0 \in \Sigma^{ch} \backslash \Sigma^{in}$ the complete asymptotic forms were constructed earlier in [4, 5]. The complete asymptotic expansion consists of series in powers of ε, the coefficients of which are derivatives of Green's functions, where the order of these derivatives increases with the power of ε.

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